*The Cross-Ratio Group of $n!$ Cremona Transformations of Order $n-3$ in Flat Space of $n-3$ Dimensions.**

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Introduction.

The binary n-ic form has as an absolute irrational invariant the cross-ratio of any four of its roots. These cross-ratios are expressible rationally in terms of any $n-3$ independent ones. If any particular system of $n-3$ independent ratios be associated with a particular order of the n roots, by varying the order of the n roots, we shall have in all $n!$ conjugate systems; these systems are expressible rationally in terms of the original system, and exactly so in terms of any system of the set. Hence arises, to speak geometrically, a group of $n!$ Cremona transformations in flat space of $n-3$ dimensions. The various Cremona groups so obtained from the various initial systems are Cremona transformations of one another. In this paper I study more closely one of the simplest of such Cremona groups.

§1.

Definition of the cross-ratio group $G_{n!}$ of Cremona transformations $F^{(a)}$.

I recall certain fundamental properties of the cross-ratio rational function $[xyzu]$ of the four independent variables $xyzu$:

$$[xyzu] = \frac{(x-z)(y-u)}{(y-z)(x-u)}.$$

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*This paper, with slight modifications, was read before the American Mathematical Society at the Buffalo meeting of the summer of 1896.

The papers of G. Kohn, "Ueber die Erweiterung eines Grundbegriffes der Geometrie der Lage" (Mathematische Annalen, vol. 46, p. 285, 1895), and "Die homogenen Coordinaten als Wurfcoordinaten" (Wiener Sitzungsberichte, vol. 104, p. 1167, 1895), have certain connections of content with the present (independent) paper.

(a) $$[xyzu] \,.\, [wxzu] = [wyzu] \,.$$

(b) $$[xyzu] = [yxuz] = [zuxy] = [uzyx] \,.$$

(c) $$[xyzx] = \infty, \quad [xyzy] = 0, \quad [xyzz] = 1; \ [\infty 0 1 u] = u \,.$$

(d) $$[xyzu] \,.\, [yxzu] = 1, \quad [xyzu] + [xzyu] = 1 \,.$$

(e) The four quantities $xyzu$ taken in all $4! = 24$ orders give rise to six, and only six, different cross-ratios, and these are expressible as linear fractional functions of (any) one of them, viz.:

$$\begin{aligned}
[xyzu] &= \lambda &&= \lambda_0(\lambda), \quad & [xzyu] &= 1 - \lambda &&= \lambda_3(\lambda),\\
[yzxu] &= \frac{\lambda - 1}{\lambda} &&= \lambda_1(\lambda), \quad & [yxzu] &= \frac{1}{\lambda} &&= \lambda_4(\lambda),\\
[zxyu] &= \frac{1}{1-\lambda} &&= \lambda_2(\lambda), \quad & [zyxu] &= \frac{\lambda}{\lambda - 1} &&= \lambda_5(\lambda).
\end{aligned}$$

(f) The cross-ratio $[xyzu]$ is invariant under cogredient linear fractional transformation

$$v' = \frac{av + b}{cv + d} \qquad (ad - bc \neq 0)$$

of its variables x, y, z, u:

$$[xyzu] = [x' y' z' u] \,.$$

(g) x, y, z being unequal, by the transformation

$$v' = \frac{(x-z)(y-v)}{(y-z)(x-v)} = [xyzv] \,,$$
$$(ad - bc = -(x-y)(x-z)(y-z) \neq 0) \,,$$

which throws $v = x, y, z$ to $v' = \infty, 0, 1$, we have

$$[xyzu] = [x' y' z' u'] = [\infty 0 1 u'] = u' \,.$$

We take now n independent variables

$$z_i \qquad (i = 1, 2, \ldots, n) \qquad (1)$$

and consider the $n(n-1)(n-2)(n-3)$ cross-ratios

$$r_{ijkl} = [z_i z_j z_k z_l] \,, \qquad (i, j, k, l = 1, 2, \ldots, n). \qquad (2)$$

Of the n ratios

$$r_i = [z_n z_{n-1} z_{n-2} z_i] \,, \qquad (i = 1, 2, \ldots, n), \qquad (3)$$

the three r_n, r_{n-1}, r_{n-2} have the numerical values ∞, 0, 1 respectively, while the $n-3$ remaining $r_i\,(i=1, 2, \ldots, n-3)$ are obviously independent and form, we say, a *fundamental system*

$$R=(r_1, r_2, \ldots, r_{n-3}) \tag{4}$$

of $n-3$ cross-ratios for the rational expression (in accordance with the remarks a, b, d) of all the cross-ratios r_{ijkl}. Indeed, by applying the transformation $v'=[z_n z_{n-1} z_{n-2} v]$ to the n variables $v=z_i$, we have (by g, f), since $r_i=z'_i$,

$$r_{ijkl}=[r_i r_j r_k r_l]. \tag{5}$$

Similarly every order a of the n variables $z_1 \ldots z_n$,

$$a=(z_{a_1} z_{a_2} \ldots z_{a_n}), \tag{6}$$

gives rise to a corresponding fundamental system

$$R^{(a)}=(r_1^{(a)}, r_2^{(a)}, \ldots, r_{n-3}^{(a)}), \quad r_i^{(a)}=[z_{a_n} z_{a_{n-1}} z_{a_{n-2}} z_{a_i}], \quad (i=1, 2, \ldots, n-3). \tag{7}$$

The two fundamental systems R, $R^{(a)}$ are, of course, expressible each in terms of the other. We write the system of $n-3$ equations

$$r_i^{(a)}=[r_{a_n} r_{a_{n-1}} r_{a_{n-2}} r_{a_i}]=f_i^{(a)}(r_1, r_2, \ldots, r_{n-3}) \quad (i=1, 2, \ldots, n-3) \tag{8}$$

more compactly

$$(r_1^{(a)}, r_2^{(a)}, \ldots, r_{n-3}^{(a)})=(f_1^{(a)}, f_2^{(a)}, \ldots, f_{n-3}^{(a)})(r_1, r_2, \ldots, r_{n-3}), \tag{8'}$$

or

$$R^{(a)}=F^{(a)} R, \tag{8''}$$

where, on the right, $F^{(a)}=(f_1^{(a)}, f_2^{(a)}, \ldots, f_{n-3}^{(a)})$ is a system of $n-3$ functional operators each on $n-3$ arguments.

Now, in a flat space R_{n-3} of $n-3$ dimensions, a point R being determined by the $n-3$ non-homogeneous point-coordinates r_i,

$$R=(r_1, r_2, \ldots, r_{n-3}), \tag{9}$$

we determine by

$$R'=F^{(a)} R, \tag{10}$$

a 1, 1 rational point-transformation; that is, a *Cremona transformation $F^{(a)}$ of the flat space R_{n-3}*. The *order* of this transformation $F^{(a)}$, that is, the order of the $(n-4)$-way locus-spread in the flat space R_{n-3} of points R which corres-

ponds to the general linear (flat) spread R_{n-4} in the flat space R_{n-3} of points R', is

$$1 \text{ if } a_n = n, \quad n-3 \text{ if } a_n \neq n.$$

For, indeed, the transformation $F^{(a)}$ (8, 8′)

$$r'_1, \ldots, r'_i, \ldots, r'_{n-3}$$
$$= \frac{r_{a_n} - r_{a_{n-2}}}{r_{a_{n-1}} - r_{a_{n-2}}} \left(\frac{r_{a_{n-1}} - r_{a_1}}{r_{a_n} - r_{a_1}}, \ldots, \frac{r_{a_{n-1}} - r_{a_i}}{r_{a_n} - r_{a_i}}, \ldots, \frac{r_{a_{n-1}} - r_{a_{n-3}}}{r_{a_n} - r_{a_{n-3}}} \right) \quad (11)$$

is for the four cases $n = a_n$, a_{n-1}, a_{n-2}, a_j $(j \leqq n-3)$, when we recall that $r_n = \infty$:

$a_n = n)$ $\quad r'_1, \ldots, r'_i, \ldots, r'_{n-3}$
$$= \frac{r_{a_{n-1}} - r_{a_1}}{r_{a_{n-1}} - r_{a_{n-2}}}, \ldots, \frac{r_{a_{n-1}} - r_{a_i}}{r_{a_{n-1}} - r_{a_{n-2}}}, \ldots, \frac{r_{a_{n-1}} - r_{a_{n-3}}}{r_{a_{n-1}} - r_{a_{n-2}}}. \quad (11_1)$$

$a_{n-1} = n)$ $\quad r'_1, \ldots, r'_i, \ldots, r'_{n-3}$
$$= \frac{r_{a_n} - r_{a_{n-2}}}{r_{a_n} - r_{a_1}}, \ldots, \frac{r_{a_n} - r_{a_{n-2}}}{r_{a_n} - r_{a_i}}, \ldots, \frac{r_{a_n} - r_{a_{n-2}}}{r_{a_n} - r_{a_{n-3}}}. \quad (11_2)$$

$a_{n-2} = n)$ $\quad r'_1, \ldots, r'_i, \ldots, n_{n-3}$
$$= \frac{r_{a_{n-1}} - r_{a_1}}{r_{a_n} - r_{a_1}}, \ldots, \frac{r_{a_{n-1}} - r_{a_i}}{r_{a_n} - r_{a_i}}, \ldots, \frac{r_{a_{n-1}} - r_{a_{n-3}}}{r_{a_n} - r_{a_{n-3}}}. \quad (11_3)$$

$a_j \quad = n)$ $\quad \ldots, r'_i, \ldots, r'_j, \ldots$
$$= \frac{r_{a_n} - r_{a_{n-2}}}{r_{a_{n-1}} - r_{a_{n-2}}} \left(\ldots, \frac{r_{a_{n-1}} - r_{a_i}}{r_{a_n} - r_{a_i}}, \ldots, 1, \ldots \right) \quad (11_4)$$
$$(i \neq j,\ i = 1, 2, \ldots, n-3).$$

The *literal substitution* α

$$\alpha = \begin{pmatrix} z_1, & \ldots, & z_i, & \ldots, & z_n \\ z_{a_1}, & \ldots, & z_{a_i}, & \ldots, & z_{a_n} \end{pmatrix} = (z_i\, z_{a_i}) \quad (12)$$

changes the initial order o to the order a. There is a *composition of substitutions*

$$\alpha\beta = \gamma, \quad (z_i\, z_{a_i})(z_j\, z_{b_j}) = (z_k\, z_{c_k}), \quad (c_k = b_{a_k} : k = 1, 2, \ldots, n), \quad (13)$$

and similarly a *composition of orders*

$$ab = c, \quad (z_{a_1} \ldots z_{a_n})(z_{b_1} \ldots z_{b_n}) = (z_{c_1} \ldots z_{c_n}) \quad (c_k = b_{a_k} : k = 1, 2, \ldots, n). \quad (14)$$

The order $c = ab$ is derived from the order b by the same *position-permutation* as the order $a = ao$ from the order o.

Hence, just as $R^{(a)} = F^{(a)} R = F^{(a)} R^{(o)}$ (8″), so $R^{(c)} = R^{(ab)} = F^{(a)} R^{(b)}$. But $R^{(b)} = F^{(b)} R^{(o)}$ and $R^{(c)} = F^{(c)} R^{(o)}$. Thus, corresponding to (13) and (14), we have

the *composition of cross-ratio transformations* $F^{(a)}$:

$$F^{(a)}F^{(b)}=F^{(ab)}, \tag{15}$$
$$R''=F^{(a)}R',\quad R'=F^{(b)}R;\quad R''=F^{(a)}(F^{(b)}R)=F^{(ab)}R.$$

The $n!$ substitutions a form the symmetric substitution-group $G^n_{n!}$ of order $n!$.

The $n!$ transformations $R'=F^{(a)}R\,(8'')$ of the flat space R_{n-3} are for $n\geqq 5$* distinct,† and form a holoedrically isomorphic transformation-group $G_{n!}$ which, from its source, I call the *cross-ratio transformation-group $G_{n!}$ of the flat space of $n-3$ dimensions R_{n-3}.*

The $(n-1)!$ collineations $F^{(a)}\,(a_n=1)$ form by themselves a collineation group $G_{(n-1)!}$ holoedrically isomorphic with the symmetric substitution-group $G^{n-1}_{(n-1)!}$ on the $n-1$ letters $z_1\,\ldots\,z_{n-1}$.

The remaining $n!-(n-1)!$ transformations $F^{(a)}$ are Cremona transformations of order $n-3$.

§2.

The collineation-group $G_{(n-1)!}$ of the transformations $F^{(a)}\,(a_n=n)$ is the (Klein's) group of $(n-1)$ collineations permuting amongst themselves in all possible ways certain $n-1$ points $P_1,\ldots,P_{n-1}$.

We introduce the homogeneous point-coordinates x,

$$R=(r_1,\ldots,r_{n-3})=(x_1:x_2:\ldots:x_{n-3}:x_{n-2}), \tag{16}$$

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*For $n=4$, the group is made up of the six linear fractional transformations $r'=\lambda_i(r)$ $(i=0,1,\ldots,5)$. This well-known group plays a fundamental rôle in the theory of the binary quartic.

The general cross-ratio group $G_{n!}$ $(n\geq 5)$ I obtained in 1895.

Mr. Slaught, in his forthcoming Chicago dissertation, discusses in detail the group for $n=5$.

The group for $n=5$ has been given (from a standpoint quite different) by Mr. S. Kantor, "Theorie der endlichen Gruppen von eindeutiger Transformationen in der Ebene" (Berlin, 1895, pp. 11, 19, 51, 52, 105). And Mr. Kantor may have somewhere indicated the generalization to $n=n$.

†If the two transformations $F^{(a)}$, $F^{(b)}$ are identical: $F^{(a)}R=F^{(b)}R$ for every point R, then denoting by a' the order reciprocal to a $(a'a=o)$ and by c the product $a'b$ one has $F^{(c)}R=F^{(o)}R$, that is, when one lets the point R depend as in (9, 3) upon the n variables z_i (1),

$$[z_{c_n}z_{c_{n-1}}z_{c_{n-2}}z_{c_i}]=[z_nz_{n-1}z_{n-2}z_i], \qquad (i=1,\ldots,n-3).$$

The z's being general variables, this implies for every i that the two tetrads $z_{c_n}z_{c_{n-1}}z_{c_{n-1}}z_{c_i}$, $z_n\,z_{n-1}z_{n-2}\,z_i$ are the same order apart, whence the fixed triads $z_{c_n}\,z_{c_{n-1}}z_{c_{n-2}}$, $z_nz_{n-1}\,z_{n-2}$ are the same and so the residual z_{c_i}, z_i. Thus $c_i=i$ for $i=1,\ldots,n-3$, and, indeed (by e), for $i=1,\ldots,n$. That is, the orders c and o are the same, and likewise the orders a and b.

where $$r_i = x_i / x_{n-2}, \qquad (i = 1, 2, \ldots, n-3). \qquad (17)$$

It is convenient to introduce also the symbols x_{n-1}, x_n with the respective values $x_{n-1} = 0$, $x_n = \infty$. Then, remembering that $r_{n-2} = 1$, $r_{n-1} = 0$, $r_n = \infty$, we have

$$r_i = x_i / x_{n-2}, \qquad (i = 1, \ldots, n). \qquad (17')$$

We introduce also the supernumerary homogeneous point-coordinates y:

$$y_i = \sum_{g=1}^{g=n-1} x_g - (n-1)\, x_i, \quad (n-1)\, x_i = y_{n-1} - y^i \qquad (18)$$
$$(i = 1, \ldots, n-1)$$

with the identity

$$\sum_{i=1}^{i=n-1} y_i \equiv 0. \qquad (19)$$

The collineation $F^{(a)}(a_n = n)$ (11_1) is then

$$\mu x_i' = x_{a_i} - x_{a_{n-1}}, \qquad (i = 1, 2, \ldots, n-2), \qquad (20)$$

or $$\mu y_i' = y_{a_i}, \qquad (i = 1, 2, \ldots, n-1), \qquad (20')$$

where μ is a proportionality-factor.

The group $G_{(n-1)!}$ of $(n-1)!$ collineations $F^{(a)}(a_n = n)$ is then Klein's group.* It permutes amongst themselves in all possible ways the $n-1$ points $P_j\,(j = 1, 2, \ldots, n-1)$:

$$(y_1 : \ldots : y_i : \ldots : y_j : \ldots : y_{n-1}) = (1 : \ldots : 1 : \ldots : -(n-2) : \ldots : 1), \qquad (21)$$
$$(x_1 : \ldots : x_i : \ldots : x_j : \ldots : x_{n-2}) = (0 : \ldots : 0 : \ldots : \quad 1 \quad : \ldots : 0), \quad (j \neq n-1),$$
$$(x_1 : \ldots : x_i : \ldots\ldots\ldots : x_{n-2}) = (1 : \ldots : 1 : \ldots\ldots\ldots : 1), \quad (j = n-1),$$

which are more conveniently given by the use of Kronecker's symbol δ_{st} with the definition:

$$\delta_{st} = 0 \ (s \neq t), \quad \delta_{st} = \delta_{ss} = 1 \ (s = t) \qquad (22)$$

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*Klein, "Ueber eine geometrische Repräsentation der Resolventen algebraischer Gleichungen" (Mathematische Annalen, vol. IV, pp. 346–358, 1871).

in the form:

For $j=1,\ldots,n-1$:

$$\mu y_i = 1-(n-1)\,\delta_{ij};\quad \mu x_i = \delta_{ij}-\delta_{n-1\,j}, \qquad (i=1,2,\ldots,n-1). \tag{21'}$$

In fact the collineation $R' = F^{(a)}R$ throws the point $R = P_{a_k}$ to the point $R' = P_k$ $(k = 1, 2, \ldots, n-1)$.

§3.

The non-linear Cremona transformations $F^{(a)}$ $(a_n \neq n)$ have critical figures included in the complete $(n-1)$-gon P_i $(i = 1, 2, \ldots, n-1)$.

In a flat space R_{m-1} of $m-1$ dimensions the simplest non-linear Cremona transformation is the well-known *inversion* (*of period two*):

$$\mu x'_i = 1/x_i, \qquad (i=1,2,\ldots,m). \tag{23}$$

Its critical points R_0 and flats R_{m-2} are the m vertices and the $\frac{1}{2}m(m-1)$ face-R_{m-2} of the complete m-gon of coordinate vertices. The general linear spread R_{m-2} of the flat space R_{m-1} of points R' corresponds to a spread $S_{m-2,\,m-1,\,m-1}$ of $m-2$ dimensions of order $m-1$ in the flat space R_{m-1} of points R having $(m-2)$-ple points at the coordinate vertices, and vice versa. The general straight line R_1 of the $R_{m-1}(R')$ corresponds to a rational curve $S_{1,\,m-1,\,m-1}$ of order $m-1$ of the $R_{m-1}(R)$ containing the coordinate vertices, and vice versa. The fixed points are the 2^{m-1} points $\rho x_i = \varepsilon_i$ where $\varepsilon_i = +1$ or -1. These come out of one of them by the group of 2^{m-1} projective reflections connected with the fundamental m-gon, viz., $\rho x'_i = \varepsilon_i x_i$ $(i = 1, 2, \ldots, m)$. The inversion is fully and uniquely determined by its m critical points and one of its fixed points, no m of which $m+1$ points lie in a flat R_{m-2}. A flat R_k $(k \leqq m-2)$ is invariant under the inversion if and only if it connects one of the fixed points to k of the critical points.

To return to our cross-ratio transformations $F^{(a)}$, we consider now any substitution α with $a_n = j \neq n$. The simplest such substitution is the transposition $\beta = (z_j z_n)$. The substitution $\gamma = \alpha\beta$ has $c_n = n$. Further, $\alpha = \gamma\beta^{-1} = \gamma\beta$.

Similarly, the most general transformation $F^{(a)}$ $(a_n = j \neq n)$ results from the composition

$$F^{(a)} = F^{(c)}\,F^{(b)} \tag{24}$$

of the *collineation* $F^{(c)}$ (§2) and the *inversion* $F^{(b)}$. The transformation

$$F^{(b)} = F^{(b)_j} \text{ where } b_j = n,\ b_n = j,\ b_i = i \qquad (25)$$
$$(i = j, n;\ j \neq n;\ i = 1, 2, \ldots, n)$$

is in fact an inversion, for the formulas $(11_{2,3,4})$ of §1 may, by the use of the notations of §2 and the properly determined coordinate system

$$x_{ij} \equiv x_i - x_j, \qquad (i = 1, 2, \ldots, n-1), \qquad (26)$$

be written

$$\mu x'_{ij} = 1/x_{ij}, \qquad (i \neq j,\ i = 1, 2, \ldots, n-1). \qquad (27)$$

This inversion $F^{(b)_j}$ has as vertices of its fundamental critical $(n-2)$-gon the $n-2$ points $P_i\,(i \neq j)$, while one of its 2^{n-3} fixed points is P_j.

For the inversion $R' = F^{(b)_j} R$ the critical figures* for the two flats $R_{n-3}(R)$, $R_{n-3}(R')$ are identical. For the general transformation $R'' = F^{(a)} R$, viz., $R'' = F^{(c)} R'$, $R' = F^{(b)_j} R$, the critical figures,* while no longer the same, are parts of the complete $(n-1)$-gon $P_i\,(i = 1, 2, \ldots, n-1)$, viz., in the flat $R_{n-3}(R)$ the complete $(n-2)$-gon $P_i\,(i \neq j)$, and in the flat $R_{n-3}(R')$ the complete $(n-2)$-gon $P_{d_i}\,(i \neq j)$, where the orders d and c are reciprocal.

The results of §§2, 3 show that the cross-ratio transformation-group $G_{n!}$ of the flat R_{n-3} is determined fully and uniquely by any system of $n-1$ linearly independent points P_i (that is, such that no $n-2$ of them lie in a flat R_{n-4}). After giving in §4 certain preliminaries, we proceed in §5 to a closer analysis of this new determination of the group.

§4.

Concerning rational curves $S_{1,\,n-3,\,n-3}$ of order $n-3$ in the flat R_{n-3}.

The general rational curve $S_{1,\,n-3,\,n-3}$ has its running point $(x_1 : \ldots : x_{n-2})$ given in terms of the parameter $\sigma = \sigma_1/\sigma_2$ by the equations

$$\mu x_i = G_i(\sigma_1, \sigma_2) \equiv \sum_{j=1}^{j=n-2} g_{ij}\, \sigma_1^{n-2-j} \sigma_2^{j-1}, \qquad (28)$$
$$(i = 1, 2, \ldots, n-2),$$

where the coefficients g_{ij} are constants with determinant $|g_{ij}|$ $(i, j = 1, 2, \ldots, n-2)$ not zero.

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* For $n=5$ these results are given by Messrs. Kantor and Slaught.

By a change of coordinate-system, this may be written

$$\mu x_i = \sigma_1^{n-2-i}\sigma_2^{i-1}, \qquad (i=1,\, 2, \ldots, n-2). \tag{29}$$

This normal form of the equation of the curve shows that just as to every parameter-value σ one point of the curve corresponds, so also to every point of the curve one parameter-value corresponds.

The flat R_{n-4}

$$\sum_{i=1}^{i=n-2} l_i x_i = 0 \tag{30}$$

meets the curve in the $n-3$ points corresponding to the $n-3$ roots $\sigma = \sigma_1/\sigma_2$ of the equation

$$\sum_{i=1}^{i=n-2} l_i \sigma_1^{n-2-i}\sigma_2^{i-1} = 0, \tag{31}$$

and, conversely, any $n-3$ points of the curve lie in one and only one flat R_{n-4}.

This normal coordinate-system is fully defined by the curve and the three points $\sigma = \infty,\ 0,\ 1$ on the curve; the $(n-4)$-flat $x_i = 0$ is *the* flat meeting the curve in the point $\sigma = 0$ $n-2-i$ times, and in the point $\sigma = \infty$ $i-1$ times; the unit-point $x_1 = \ldots. = x_{n-2}$ is the point $\sigma = 1$.

By a linear homogeneous substitution on the σ_1, σ_2, i. e., by a linear fractional substitution on the parameter σ, we change the distribution of the parameter-values over the points of the curve. There are ∞^3 such distributions and correspondingly ∞^3 normal coordinate-systems.

If a curve is given twice, once with the parameter σ and once with the parameter τ, then

$$\sigma = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \qquad (\alpha\delta-\beta\gamma \neq 0), \tag{32}$$

i. e., the ∞^3 distributions of the preceding remark exhaust all possible distributions. For, at least, we can so determine $\alpha\beta\gamma\delta$ that τ', where

$$\tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta},$$

shall be ∞, 0, 1 at the same points at which σ is ∞, 0, 1. Then the two distributions σ, τ' lead to the same normal coordinate-system, and hence, indeed, $\sigma = \tau'$.

Hence for *every* parameter-distribution the cross-ratio of the parameters of any certain four points of the curve is the same, and we speak of the *cross-ratio of four points of the curve.*

By a collineation of the R_{n-3}, which throws one normal coordinate-system into another, the curve is thrown into itself; on the curve, the point $\sigma = \sigma_0$ (with respect to a *fixed* parameter-distribution) is thrown to the point $\sigma = \sigma_0'$, where $\sigma_0' = (\alpha\sigma_0 + \beta)/(\gamma\sigma_0 + \delta)$. These ∞^3 collineations of the curve into itself are the only* ones.

Through any system of n linearly independent points Q_j ($j = 1, 2, \ldots, n$) of the R_{n-3} one and only one rational curve $S_{1,\,n-3,\,n-3}$ passes. By proper determination of the coordinate-system, we may set:

$$Q_j: \quad x_1 : \ldots : x_i : \ldots : x_j : \ldots : x_{n-2} = 0 : \ldots : 0 : \ldots : 1 : \ldots : 0, \quad (j = 1, 2, \ldots, n-2), \tag{33}$$

$$Q_{n-1}: \quad x_1 : \ldots : x_i : \ldots\ldots : x_{n-2} = 1 : \ldots : 1 : \ldots\ldots : 1,$$

$$Q_n: \quad x_1 : \ldots : x_i : \ldots\ldots : x_{n-2} = \xi_1 : \ldots : \xi_i : \ldots\ldots : \xi_{n-2},$$

where no ξ is 0 and no two ξ's are equal. Let the parameter σ on the rational curve take the value $\sigma = \lambda_j$ at Q_j ($j = 1, 2, \ldots, n$). We fix the parameter σ by the conditions $\lambda_n = \infty$, $\lambda_{n-1} = 0$, $\lambda_{n-2} = 1$. Setting

$$G(\sigma_1, \sigma_2) = \prod_{j=1}^{j=n-2} (\sigma_1 - \lambda_j \sigma_2), \quad G_i(\sigma_1, \sigma_2) = g_i\, G(\sigma_1, \sigma_2)/(\sigma_1 - \lambda_i \sigma_2), \quad (i = 1, 2, \ldots, n-2), \tag{34}$$

where the g_i's are constants, we have as parametric representation of the most general rational curve $S_{1,\,n-3,\,n-3}$ containing the points Q_j with the respective parametric values $\sigma = \lambda_j$ ($j = 1, 2, \ldots, n-2$),

$$\mu x_i = G_i(\sigma_1, \sigma_2), \qquad (i = 1, 2, \ldots, n-2). \tag{35}$$

The condition that the curve-point $\sigma = \lambda_{n-1} = 0$ shall be Q_{n-1} is $\mu' g_i = \lambda_i$ ($i = 1, 2, \ldots, n-2$), and the condition that the curve-point $\sigma = \lambda_n = \infty$ shall be Q_n is $\mu'' g_i = \xi_i$ ($i = 1, 2, \ldots, n-2$). We set then in (34, 35)

$$g_i = \xi_i, \quad \lambda_i = \xi_i/\xi_{n-2}, \qquad (i = 1, 2, \ldots, n-2), \tag{36}$$

and have, indeed, in (35) *the single rational curve* $S_{1,\,n-3,\,n-3}$ which contains the n points Q_j ($j = 1, 2, \ldots, n$). (That the determinant $|g_{ij}| \neq 0$ is a consequence of the linear independence of the points Q_j.)

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* This theorem is given, for instance, by Meyer, "Apolarität und rationale Curven," p. 398, 1883.

§5.

The cross-ratio transformation group $G_{n!}\{F^{(a)}\}$ is thus projectively determined by its fundamental $(n-1)$-gon P_j $(j=1, 2, .., n-1)$:

If P_n is any point linearly independent of the P_j $(j=1, 2, .., n-1)$, then the transformation $F^{(a)}$, $R'=F^{(a)}R$, throws the point $R=P_n$ to the point $R'=F^{(a)}P_n$ such that the two n-gons

$$P_1 P_2 .. P_{n-1} F^{(a)} P_n, \quad P_{a_1} P_{a_2} .. P_{a_{n-1}} P_{a_n}$$

are projective.

We take any system of n linearly independent points P_i $(i=1, 2, .., n)$ of the flat R_{n-3}. Through them one and only one rational curve $S_{1,\,n-3,\,n-3}$ passes (§4). Identifying them taken in any order $a=(P_{a_1} .. P_{a_n})$ with the n points $(Q_1 .. Q_n)$ of §4, $P_{a_i}=Q_i=P_i^{(a)}$, we determine the homogeneous coordinate system $(x_1 : .... : x_{n-2})$ of §4, or say $(x_1^{(a)} : .. : x_{n-2}^{(a)})$, and the corresponding non-homogeneous system $(r_1^{(a)}, .., r_{n-3}^{(a)})$ where $r_i^{(a)}=x_i^{(a)}/x_{n-2}^{(a)}$ $(i=1, 2, .., n-2)$. Then P_{a_n} is say

$$P_{a_n}: \quad (x_1^{(a)} : .. : x_{n-2}^{(a)}=\xi_1^{(a)} : .. : \xi_{n-2}^{(a)}), \quad (r_1^{(a)}, .., r_{n-3}^{(a)}=\rho_1^{(a)}, .., \rho_{n-3}^{(a)}). \tag{37}$$

The parameter $\sigma^{(a)}$ of the curve shall take at P_{a_i} the value $\sigma^{(a)}=\lambda_i^{(a)}$ and in particular at P_{a_n}, $P_{a_{n-1}}$, $P_{a_{n-2}}$ the respective values $\lambda_n^{(a)}=\infty$, $\lambda_{n-1}^{(a)}=0$, $\lambda_{n-2}^{(a)}=1$. Then since by §4 (36), $\lambda_i^{(a)}=\xi_i^{(a)}/\xi_{n-2}^{(a)}$ $(i=1, 2, .., n-2)$, we have

$$\lambda_i^{(a)}=\rho_i^{(a)}, \qquad (i=1, 2, .., n-2). \tag{38}$$

Further by (4),

$$\lambda_i^{(a)}=[\infty 0 1 \lambda_i^{(a)}]=[\lambda_n^{(a)} \lambda_{n-1}^{(a)} \lambda_{n-2}^{(a)} \lambda_i^{(a)}]=[P_{a_n} P_{a_{n-1}} P_{a_{n-2}} P_{a_i}]_{\text{(on curve)}}. \tag{39}$$

We now refer everything to the original order $o=(P_1 .. P_n)$, and drop the (o) from the x_i, r_i, ξ_i, ρ_i, σ_i, λ_i. We further identify the $n-1$ points $P_1 .. P_{n-1}$ with the $n-1$ fundamental points $P_1 .. P_{n-1}$ of the cross-ratio transformation group $G_{n!}$ (§2).

The $x_i^{(a)}$'s are expressed in the x_j's identically thus:

$$\mu x_i^{(a)} \equiv \sum_{j=1}^{j=n-2} q_{ij}^{(a)} x_j, \qquad (i=1, .., n-2), \tag{40}$$

where μ is a proportionality factor and the $q_{ij}^{(a)}$'s are constants whose determinant is not 0,

The collineation

$$\mu x_i' = \mu x_i^{(a)} = \sum_{j=1}^{j=n-2} q_{ij}^{(a)} x_j, \quad r_i' = r_i^{(a)}, \qquad (i=1, 2, \ldots, n-2) \tag{41}$$

throws P_{a_i} *to* $P_i\,(i=1,\ 2,\ldots,\ n-1)$, *and,* as I shall prove, P_{a_n} *to* $F^{(a)}P_n$.

In fact, since (§4) the cross-ratio of four points on the curve is independent of the choice of parameter, we have

$$\lambda_i^{(a)} = [P_{a_n} P_{a_{n-1}} P_{a_{n-2}} P_{a_i}]_{\text{(on curve)}} = [\lambda_{a_n} \lambda_{a_{n-1}} \lambda_{a_{n-2}} \lambda_{a_i}] = f_i^{(a)}(\lambda_1, \lambda_2, \ldots, \lambda_{n-3}), \qquad (i=1, 2, \ldots, n-3). \tag{42}$$

Further, since $\lambda_i^{(a)} = \rho_i^{(a)}$ and $\lambda_i = \rho_i$, the collineation (41):

$$r_i' = r_i^{(a)}, \qquad (i=1,\ 2,\ldots,\ n-3), \tag{41}$$

does indeed throw the point

$$R = P_{a_n}: \qquad r_i^{(a)} = \rho_i^{(a)} \qquad (i=1,\ 2,\ldots,\ n-3) \tag{43}$$

to the point

$$R' = F^{(a)}P_n: \qquad r_i' = f_i^{(a)}(\rho_1, \rho_2, \ldots, \rho_{n-3}), \qquad (i=1, 2, \ldots, n-3). \tag{44}$$

§6.

The fixed points of the transformations $F^{(a)}$ and binary n-ic forms with collineations into themselves.

We consider (only) the fixed points which are linearly independent of the $n-1$ fundamental points $P_1 \ldots P_{n-1}$ of the group $G_{n!}$.

If $R = P_n$ is such a fixed point for the transformation $F^{(a)}$, then (§5) we have the projectivity

$$[P_1 P_2 \ldots P_{n-1} P_n] \barwedge [P_{a_1} P_{a_2} \ldots P_{a_{n-1}} P_{a_n}]. \tag{45}$$

The collineation (41) throws the n-gon into itself, and hence the rational curve $S_{1,\,n-3,\,n-3}$ through the n points P_i into itself, and on the curve is equivalent to a collineation of the fundamental parameter σ. To the n points P_i corresponds then a binary n-ic form in $\sigma_1, \sigma_2\ (\sigma = \sigma_1/\sigma_2)$ with a collineation into itself.

Conversely, to every binary n-ic form $H(\sigma_1, \sigma_2)$ $(n > 4)$ of non-vanishing discriminant invariant (up to a constant factor) under a group of d binary collineations G_d', where then d is a factor of $n!$, corresponds $n!/d$ points P_n in the R_{n-3} each invariant under a subgroup G_d of d transformations $F^{(a)}$ of the cross-

ratio group $G_{n!}$; these $n!/d$ points and subgroups are conjugate under the main group. We determine these points P_n by linearly transforming $H(\sigma_1, \sigma_2)$ in all possible ways into a *normal form* $\overline{H}(\overline{\sigma_1}, \overline{\sigma_2})$ with the factor $\overline{\sigma_1}, \overline{\sigma_2}(\overline{\sigma_1}-\overline{\sigma_2})$. If $\overline{H}(\overline{\sigma_1}, \overline{\sigma_2}) \equiv \overline{\sigma_1}\,\overline{\sigma_2}(\overline{\sigma_1}-\overline{\sigma_2}) \prod_{i=1}^{i=n-3} (\overline{\sigma_1}-\lambda_i \overline{\sigma_2})$, then the point $\mu_i = \lambda_i\, (i=1, 2, \ldots, n-3)$ is such a point P_n.

Obviously if the λ_i's of the normal form $\overline{H}(\overline{\sigma_1}, \overline{\sigma_2})$ depend upon certain t arbitrary parameters $\theta_1, \ldots, \theta_t\, (t < n-3)$, then the locus of the point $P_n(\theta_1, \ldots, \theta_t)$ is a t-fold spread fixed by points under the corresponding group G_d of d transformations $F^{(a)}$.

Now Klein (e. g., in his "Vorlesungen über das Ikosaeder . . . ," 1884) has determined all finite groups of binary collineations and the corresponding systems of invariant binary forms. In order to apply the preceding theory for any particular value of n, it is then comparatively easy to determine* the binary n-ic forms of non-vanishing discriminant with collineations into themselves.

If we consider only the collineations $F^{(a)}\,(a_n = n)$, we have in this §6 results connecting the fixed points of Klein's group of $(n-1)!$ collineations in R_{n-3} with the binary n-ic forms invariant under binary collineations for each of which one of the fixed points is a zero point of the n-ic form.

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* See, for instance, Bolza, "On Binary Sextics with Linear Transformations into Themselves" (American Journal of Mathematics, vol. 10, pp. 47–70).

For $n = 5$ the results may be compared with those obtained directly by Mr. Slaught.